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STUDY OF FORCED CONVECTION IN THE PRESENCE OF A LIQUID–POROUS-MEDIUM INTERFACE

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Exact solutions of the problem of forced convection of an incompressible fluid (gas) are obtained for a number of geometries with the presence of an interface between an "ideal" fluid and a saturated porous medium. A generalization of the Darcy law allowing for viscous effects that are substantial at the boundaries of a porous body is used to model flow in a porous layer. The flow velocity as a function of the Darcy number is studied.

Porous materials find application in many branches of modern industry. Heat pipes, thermal insulation, and geothermal installations are a far from complete list of examples of this application. An adequate description of systems with porous media is also important for modeling processes of ingot solidification, since the two-phase zone formed in solidification of alloys of noneutectic composition can be considered a medium with variable porosity.

The Darcy law traditionally used for modeling flows in porous media gives rather inaccurate results in a number of important applications. One of the generalizations of the Darcy law is made in [1], where it is suggested to additionally introduce a viscous term, thus making it possible to allow for the effects arising at the boundaries of a porous body. Another generalization of the Darcy law was made in [2], which proposed allowance for inertia effects, which are significant at large rates of filtration, by introducing a term proportional to the square of the filtration rate. Since then the influence of viscous and inertia effects has been studied in many works, e.g., [3-9]. In general, the equation describing the steady-state flow of an incompressible fluid (gas) in a porous medium can be presented as

$$-\frac{dp^*}{dx^*} + \mu_f \frac{d^2 u^*}{dy^{*2}} - \frac{\mu_f}{K} u^* - \frac{\rho_f \varepsilon F}{\sqrt{K}} u^{*2} = 0 . \tag{1}$$

The first term on the LHS of equation (1) describes the effect of the pressure gradient, the second term allows for viscous effects and makes it possible to analyze the flow at the boundaries, the third term is the traditional one following from the Darcy law, and the fourth term takes into account inertia effects arising at large rates of filtration.

The present paper deals with forced convection in the presence of an interface between a fluid and a saturated porous medium. Such a problem was first analyzed in [10] on the basis of the generalized Darcy law in the form of (1). The authors, using the perturbation technique, studied fluid flow and heat transfer at the interface of a porous medium for the following three geometries: two semi-bounded porous bodies, a semi-bounded porous body and a fluid layer, and a semi-bounded porous body with an impermeable boundary.

The first attempt to obtain an exact solution for steady-state fluid flow in a porous medium with constant porosity in the presence of an interface was undertaken in [11], where the interface between a semi-bounded porous body and a layer of "ideal" fluid was considered.

At relatively low rates of filtration the last (inertial) term in equation (1) can be ignored without substantial loss of accuracy [12-14]. Proceeding from the above assumption we seek analytical solutions describing fluid flow for a number of practically important geometries, which include an interface between a porous medium and "ideal" fluid.

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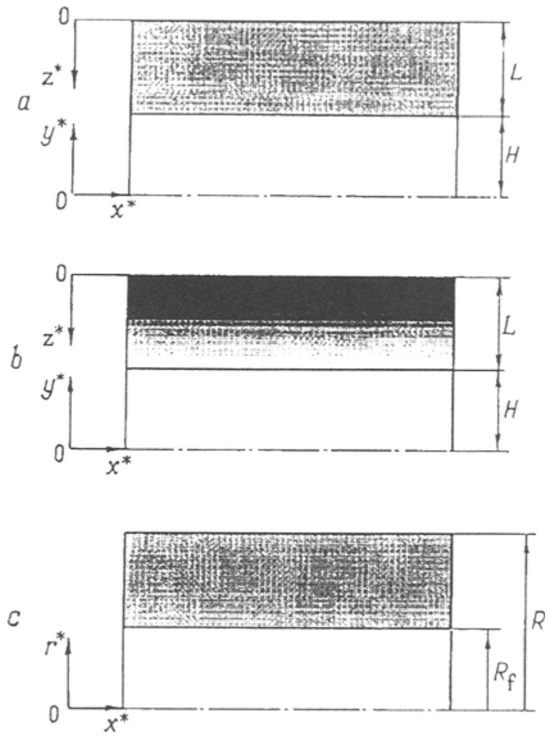

Fig. 1. Calculational schemes: a) plane channel, constant porosity; b) plane channel, variable porosity; c) cylindrical channel, constant porosity.

The three studied cases are given in Fig. 1. Case a involves steady-state flow in a plane channel along whose walls a porous layer of constant porosity is present and in the center between these two layers there is a layer of "ideal" fluid. In case b a plane channel is also analyzed, however the porosity of the porous layers is assumed to be a power function of the coordinate. And, finally, in case c a cylindrical channel is considered with the porosity assumed to be constant.

We note that one of the examples of the origination of such a flow in real systems is the familiar problem of overfreezing of a gating system in casting. If an alloy having a wide range of parameters is cast, a two-phase layer consisting of growing dendrites and liquid metal can form along the walls of the gating channels. The formation of this layer can considerably decrease metal consumption in casting, thus resulting in a number of engineering flaws.

Plane Channel, Constant Porosity. In this case fluid flow in the central layer is described by the equation

$$-\frac{dp^*}{dx^*} + \mu_f \frac{d^2u^*}{dy^{*2}} = 0 , \tag{2}$$

and fluid flow in the porous layer by

$$-\frac{dp^*}{dx^*} + \mu_f \frac{d^2u}{dz^{*2}} - \frac{\mu_f}{K} u^* = 0 . \tag{3}$$

The boundary conditions for the problem have the form

$$\left.\frac{du^*}{dy^*}\right|_{y^*=0} = 0 , \tag{4}$$

$$u^*|_{z^*=0} = 0 , \tag{5}$$

$$u^*|_{y^*=H} = u^*|_{z^*=L}, \tag{6}$$

$$\left.\frac{du^*}{dy^*}\right|_{y^*=H} = -\left.\frac{du^*}{dz^*}\right|_{z^*=L}. \tag{7}$$

The boundary condition (5) denotes the assumption about the absence of slip at the interface between a porous medium and an impermeable wall, and (6) and (7) are the conditions of "joining" of the solution at the interface between a porous medium and a layer of "ideal" fluid. These conditions and their physical meaning are discussed in [7, 11].

On introducing dimensionless variables according to the relations

$$u = \frac{u^*}{U}, \quad y = \frac{y^*}{H} \quad \text{and} \quad z = \frac{z^*}{H}, \tag{8}$$

where U is an arbitrary constant velocity, chosen for reasons of convenient normalization, Eq. (2) takes the form

$$\frac{d^2u}{dy^2} = -\Xi, \tag{9}$$

and Eq. (3) becomes

$$\frac{d^2u}{dz^2} = -\Xi + \frac{1}{\mathrm{Da}}u, \tag{10}$$

where

$$\Xi = -\frac{H^2}{\mu_f U}\frac{dp^*}{dx^*}, \quad \mathrm{Da} = \frac{K}{H^2}. \tag{11}$$

The exact solution of Eqs. (9) and (10) with the boundary conditions (4)-(7) (presentation of boundary conditions in dimensionless form is trivial) has the form

$$u = u_i + \frac{\Xi}{2}(1 - y^2) \quad \text{when} \quad 0 \le y \le 1 \quad \left(\text{which corresponds to} \ \frac{L}{H} \le z \le \frac{L}{H} + 1\right), \tag{12}$$

$$u = A_1 \exp\left(-\frac{z}{\sqrt{\mathrm{Da}}}\right) + B_1 \exp\left(\frac{z}{\sqrt{\mathrm{Da}}}\right) + \Xi\,\mathrm{Da} \quad \text{when} \quad 0 \le z \le \frac{L}{H}, \tag{13}$$

where the coordinates y and z are related by $z = -y + 1 + L/H$,

$$A_1 = -\Xi\,\mathrm{Da} - \frac{\Xi\left[\sqrt{\mathrm{Da}} - \mathrm{Da}\exp\left(-\frac{L}{H\sqrt{\mathrm{Da}}}\right)\right]}{\exp\left(-\frac{L}{H\sqrt{\mathrm{Da}}}\right) + \exp\left(\frac{L}{H\sqrt{\mathrm{Da}}}\right)},$$

$$B_1 = \frac{\Xi\left[\sqrt{\mathrm{Da}} - \mathrm{Da}\exp\left(-\frac{L}{H\sqrt{\mathrm{Da}}}\right)\right]}{\exp\left(-\frac{L}{H\sqrt{\mathrm{Da}}}\right) + \exp\left(\frac{L}{H\sqrt{\mathrm{Da}}}\right)},$$

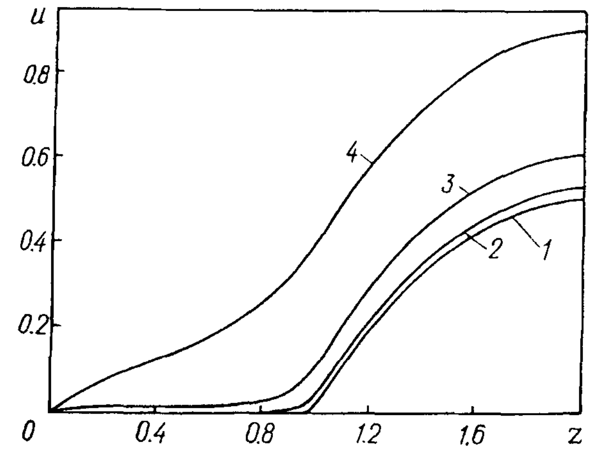

Fig. 2. Fluid flow velocity profiles for the case shown in Fig. 1a: Da = 10^{-4} (1); 10^{-3} (2); 10^{-2} (3); 10^{-1} (4).

and the flow velocity at the interface between a porous medium and "ideal" fluid u_i is found from Eq. (13) at $z = L/H$.

Fluid flow velocity profiles calculated by Eqs. (12), (13) at $L/H = 1$ and different values of Da are shown in Fig. 2. The dependence of the flow velocity on the Darcy number is vividly seen. At small of values Da (Da = 10^{-4} and Da = 10^{-3}) the velocity at the interface between a porous medium and "ideal" fluid (this interface takes place at $z = L/H = 1$) is practically zero. It increases with the Darcy number, which corresponds to a higher permeability of the porous medium, e.g., a greater porosity.

The three regions with different flow regimes in the porous layer are readily seen on the curve that corresponds to Da = 10^{-2}. Near the boundary $z = 1$ there fast retardation of the flow up to some constant velocity corresponding to fluid filtration at a distance from the boundaries (filtration described by the "classical" Darcy law). Near the boundary $z = 0$ the flow velocity continues to retard to zero, which is caused by the absence of slip at the boundary $z = 0$. Thus, at Da = 10^{-2} two boundary portions can be distinguished in the porous layer: one adjacent to the interface between the porous medium and "ideal" fluid and the other to the impermeable boundary. The rate of filtration is constant between these boundary regions. The same flow structure in the porous layer takes place at Da = 10^{-3} and Da = 10^{-4}, however, due to the smallness of the filtration rate it is not so vivid. At Da = 10^{-1} one fails to distinguish a region with a constant rate of filtration, since the regions formed at the boundaries $z = 1$ and $z = 0$ overlap. This is due to an increase in the width of these regions with the Darcy number.

Plane Channel, Variable Porosity. The channel flow shown in Fig. 1b is considered. The mathematical formulation of the problem is also determined by Eqs. (9) and (10) and boundary conditions (4)-(7), however, in this case the Darcy number entering into Eq. (10) is assumed to be a quadratic function of the coordinate z

$$\mathrm{Da} = cz^2 , \tag{14}$$

where c is a positive constant.

According to Eq. (14) the Darcy number changes from zero at $z = 0$ (which corresponds to zero permeability of medium or, in other words, to zero porosity) to some value $\mathrm{Da}_i = c(L/H)^2$ equal to the Darcy number at the interface between a porous medium and "ideal" fluid. The solution of Eq. (9) in this case still has the form of (12), and (10) is reduced to the Euler equation, and its solution [15] with allowance for boundary conditions (5)-(7) has the form

$$u = A_2 z^{(1+\sqrt{1+4/c})/2} - \frac{\Xi c}{2c - 1} z^2 , \tag{15}$$

where

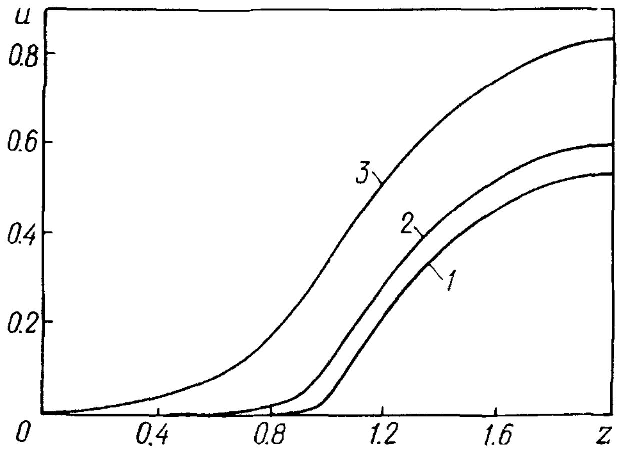

Fig. 3. Fluid flow velocity profiles for the case shown in Fig. 1b: $\mathrm{Da_i} = 10^{-3}$ (1); 10^{-2} (2); 10^{-1} (3).

$$A_2 = \frac{2\Xi\left[\frac{2c}{2c-1}\frac{L}{H}+1\right]}{(1+\sqrt{1+4/c})}\left(\frac{L}{H}\right)^{(1-\sqrt{1+4/c})/2}.$$

Equation (15) determines the velocity of fluid flow in a porous layer for $0 \le z \le L/H$.

The dependences of the fluid flow velocity on the coordinate z calculated by Eqs. (12) and (15) at $L/H = 1$ and different Da (which is equivalent to assignment of various values of the parameter c in Eq. (14)) are shown in Fig. 3. It is seen that in contrast to Fig. 2 fluid flow retardation occurs much more rapidly with "penetration" into a porous layer, and a region with a constant filtration rate cannot be distinguished on any curve. This is due to the fact that in this case, with reduction of the coordinate z, the medium's permeability gradually decreases to zero.

Cylindrical Channel, Constant Porosity. We consider the cylindrical channel shown in Fig. 1c. Instead of Eqs. (9) and (10), which are for a plane channel we should write the equations of fluid flow in radial coordinates. The fluid flow in the center of a channel free of a porous medium is described by the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = -\Xi, \tag{16}$$

and the fluid flow in the channel region occupied by a porous medium is determined by the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = -\Xi + \frac{1}{\mathrm{Da}}u, \tag{17}$$

where the dimensionless radial coordinate is determined by the expression $r = r^*/R_f$.

The boundary conditions for Eqs. (16) and (17) are similar to (4)-(7) and in dimensionless form they can be presented as

$$\left.\frac{du}{dr}\right|_{r=0} = 0, \tag{18}$$

$$u\big|_{r=R/R_f} = 0, \tag{19}$$

$$u\big|_{r=1+0} = u\big|_{r=1-0}, \tag{20}$$

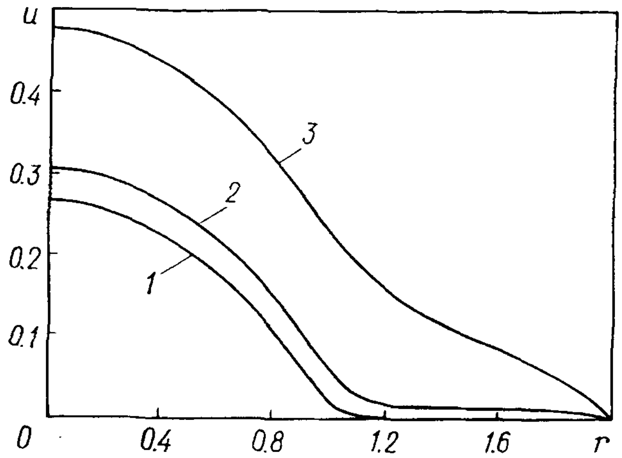

Fig. 4. Fluid flow velocity profiles for the case shown in Fig. 1c: Da = 10^{-3} (1); 10^{-2} (2); 10^{-1} (3).

$$\left.\frac{du}{dr}\right|_{r=1+0} = \left.\frac{du}{dr}\right|_{r=1-0}, \tag{21}$$

The exact solution of Eqs. (16) and (17) with boundfary conditions (18)-(21) has the form [15]

$$u = u_i + \frac{\Xi}{4}(1 - r^2) \quad \text{when} \quad 0 \le r \le 1, \tag{22}$$

$$u = A_3 I_0\left(\frac{r}{\sqrt{\mathrm{Da}}}\right) + B_3 K_0\left(\frac{r}{\sqrt{\mathrm{Da}}}\right) + \Xi\,\mathrm{Da} \quad \text{when} \quad 1 \le r \le R/R_f, \tag{23}$$

where

$$A_3 = -\Xi\frac{\mathrm{Da}}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)} - \Xi\frac{K_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}\left(\frac{\sqrt{\mathrm{Da}}}{2} - \mathrm{Da}\frac{I_1\left(\dfrac{1}{\sqrt{\mathrm{Da}}}\right)}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}\right) \times$$

$$\times\left\{K_1\left(\frac{1}{\sqrt{\mathrm{Da}}}\right) + \frac{K_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)} I_1\left(\frac{1}{\sqrt{\mathrm{Da}}}\right)\right\}^{-1},$$

$$B_3 = \Xi\left(\frac{\sqrt{\mathrm{Da}}}{2} - \mathrm{Da}\frac{I_1\left(\dfrac{1}{\sqrt{\mathrm{Da}}}\right)}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}\right)\left\{K_1\left(\frac{1}{\sqrt{\mathrm{Da}}}\right) + \frac{K_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)}{I_0\left(\dfrac{R}{R_f\sqrt{\mathrm{Da}}}\right)} I_1\left(\frac{1}{\sqrt{\mathrm{Da}}}\right)\right\}^{-1}.$$

The flow velocity at the interface between a porous medium and "ideal" fluid u_i which enters into Eq. (22) is found from (23) at $r = 1$.

The dependences of the fluid flow velocity on the radius r calculated by Eqs. (22) and (23) at $R/R_f = 2$ and different Da are shown in Fig. 4. As in Fig. 2, on the curve corresponding to Da = 10^{-2}, two boundary regions are easily seen in the porous layer, one of which is adjacent to the boundary $r = 1$ and the other to the boundary $r = 2$. The flow velocity is constant between these regions.

Conclusions. It is found that at constant porosity two boundary regions can be distinguished in a porous layer, one of which is adjacent to the boundary between the "ideal" fluid and porous medium and the other to the impermeable boundary. The filtration rate is constant between these boundary regions. Their width increases with the Darcy law; therefore, with a high permeability of the medium these regions can overlap. If the above takes place, the porous layer does not have the region with a constant rate of filtration.

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NOTATION

Da, Darcy number; $\mathrm{Da_i}$, Darcy number at the interface between "ideal" fluid and porous medium; F, dimensionless coefficient allowing for inertial effects in fluid filtration, which depends on the Reynolds number and microstructure of porous medium; I_ν and K_ν, modified Bessel functions of order ν; K, porous-medium permeability, $\mathrm{m^2}$; H, half-width of fluid layer in plane channel, m; L, width of porous layer in plane channel, m; p^*, pressure, Pa; r, dimensionless radial coordinate; r^*, radial coordinate, m; R_f, radius of interface between "ideal" fluid and porous medium in cylindrical channel, m; R, outer radius of cylindrical channel; u, velocity, m/sec; u^*, dimensionless velocity; x, y, dimensionless Cartesian coordinates; x^*, y^*, z^*, Cartesian coordinates, m; ε, porosity; μ_f, dynamic viscosity of fluid, Pa·sec; ρ_f, fluid density, $\mathrm{kg/m^3}$.

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